Epidemic Interference in a Cellular System

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Abstract

Mathematical Modelling

This article presents a mathematical modelling for the effect of interference buildup caused by a sudden increase in the number of users that access a cellular communication system

Increase in Power

For such an increase in power, a heavy tail probability distribution is generated, producing an expected increase in the error probability during the adaptation process

Stochastic Integration

The process is non-stationary, because of the sudden increase in telephone traffic, and stochastic integration can be used to attack the problem

Introduction

Introduction

- It is usual to assume stationarity in the analysis of the effect of noise or interference in communication systems, because a non-stationary environment would complicate matters in terms of mathematical modelling
- In cellular communication systems the interference is dependent on the state of the network. For example, if the users access the network early morning the traffic is clearly non-stationary, and so is the interference
- For non-stationary processes, such as in the case of a sudden increase in telephone traffic, the usual linear tools, such as basic stochastic processes, are less useful
- Stochastic integration can be used to attack the problem, based on a stochastic differential equation formulation developed by the mathematician Kazutoshi Itô

Introduction

Introduction

- The stochastic processes related to a cellular communications systems that enters an epidemic state, when most users rush to place calls, are non-stationary, which means that their statistical averages vary with time
- This makes it difficult to use the traditional correlation analysis to treat them. Stochastic integration is a useful tool to attack the problem, from the modelling of the interference based on a stochastic differential equation
- The stochastic calculus began with the study and modelling of market prices, this is, the fluctuation of the stock value as a function of time. In this case, the investors work based on the variation of the potential gain or loss, $dX(t)$, as a proportion of the invested sum $X(t)$

Introduction

Introduction

- In the present case, in fact, what matters is the relative instantaneous power, $dP(t)/P(t)$, of a certain signal, as it reacts to the channel fluctuations
- This means that the power variation should be proportional to a Wiener process $W(t)$, which is a function of the interference in the channel, that is, it increases with the rise in interference, as proved by Brzezniak and Zastawniak

$$dP(t) = \beta P(t) dW(t) \tag{1}$$

which is an informal manner to express the corresponding integral equation,

$$P(t+\tau) - P(t) = \beta \int_{t}^{t+\tau} P(u) dW(u) \tag{2}$$

Introduction

Introduction

- An immediate question associated to the equation solution is related to the non-differentiability of a Wiener process $W(t)$, at any point in time
- A way to circle the problem has been found, and is known as the theory of stochastic integrals, or the study of stochastic differential equations developed by Itô
- The Itô general stochastic equation is given by

$$dP(t) = a[P(t), t]dt + b[P(t), t]dW(t), \tag{3}$$

in which $a[P(t), t]$ is the drift function, or model trend, and $b[P(t), t]$ is the dispersion function, or volatility, of the stochastic process

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- When formulating a stochastic model to represent the interference variation along with time, it is important to consider that the interference increase rate is proportional to the existing amount of interference
- Because, as in an epidemic, the interference grows at a rate that is proportional to the number of users.
- This can be expressed, as expressed by

$$\frac{dP(t)}{dt} = \alpha P(t) \qquad (4)$$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- On the other hand, the incremental variation of the interference is proportional to the differential variation $dW(t)$ of a stochastic process, $W(t)$, which usually has a Gaussian distribution, multiplied by the total power of the active users, $P(t)$, and adjusted by the parameter β, which remains to be found, based on the channel specifications

$$dP(t) = \beta P(t) dW(t) \tag{5}$$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- The process $W(t)$ is a combination of N interference factors, $W_i(t)$, which can be found in the cellular communication system

$$W(t) = \sum_{i=1}^{N} W_i(t) \tag{6}$$

- Thus, by the Central Limit Theorem, $W(t)$ has a Gaussian probability distribution, with mean and variance given by

$$m_W(t) = \mathrm{E}[W(t)] = \sum_{i=1}^{N} \mathrm{E}[W_i(t)] \tag{7}$$

$$\sigma_W^2(t) = \mathrm{V}[W(t)] = \sum_{i=1}^{N} \mathrm{V}[W_i(t)] \tag{8}$$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- For stationary stochastic processes, or for short instants of time, the moments are independent of time, this is, $m_W(t) = Nm_W$ and $\sigma^2_W(t) = N\sigma^2_W$
- Combining both assumptions in Itô's general Equation 3, results in the following stochastic differential equation

$$dP(t) = \alpha P(t)dt + \beta P(t)dW(t) \tag{9}$$

- Which is the informal manner to write

$$P(t+\tau) = P(t)+\alpha \int_t^{t+\tau} P(u)du+\beta \int_t^{t+\tau} P(u)dW(u) \tag{10}$$

in which $P(T+\tau)$ represents the future value of the power process $P(t)$, that is τ time units in advance

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- In order to solve the stochastic differential equation obtained with the interference model, using the generic formulation for stochastic differential equations, one must consider that the drift function is proportional to the stochastic process

$$a[P(t), t] = \alpha P(t)$$

- Also, the dispersion function is modelled as

$$b[P(t), t] = \beta P(t)$$

and that one can use the Itô formula for the logarithm function $f(x, t) = \log(x)$, as shown in Itô

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- Following the procedure established by Itô, for the solution of the equation, one obtains

$$\frac{\partial f(x,t)}{\partial t} = 0 \tag{11}$$

$$\frac{\partial f(x,t)}{\partial x} = \frac{1}{x} \tag{12}$$

$$\frac{\partial^2 f(x,t)}{\partial^2 x} = -\frac{1}{x^2} \tag{13}$$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- Therefore

$$d\log[P(t)] = \frac{\alpha P(t)dt}{P(t)} + \frac{\beta P(t)dW(t)}{P(t)} - \frac{\beta^2 X^2(t)dt}{2X^2(t)} \quad (14)$$

- Which gives

$$d\log[P(t)] = \left(\alpha - \frac{\beta^2}{2}\right) dt + \beta dW(t) \quad (15)$$

- Integrating in time, one obtains

$$\log[P(t)] = \log[P(0)] + \left(\alpha - \frac{\beta^2}{2}\right) t + \beta W(t) \quad (16)$$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- Taking the logarithm inverse function, results in

$$P(t) = P(0) \cdot \exp\left(\alpha - \frac{\beta^2}{2}\right) t \cdot \exp(\beta W(t)) \qquad (17)$$

in which $P(0)$ represents the initial value of the process, that remains to be found, based on the problem constraints

- The preceding formula can be written as

$$P(t) = P(0) \cdot e^{(\alpha - \frac{\beta^2}{2})t + \beta W(t)} \qquad (18)$$

- The interference power grows exponentially in time, controlled by α and β, that remain to be determined
- It is possible to note a random variation on the curve, as a result of the stochastic process $W(t)$

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- For a Gaussian stochastic process, $W(t)$, that corresponds to the total interference, it is possible to compute the associated interference distribution, using the transformation of probability density function
- The expected value of the random power, $P(t)$, is given by

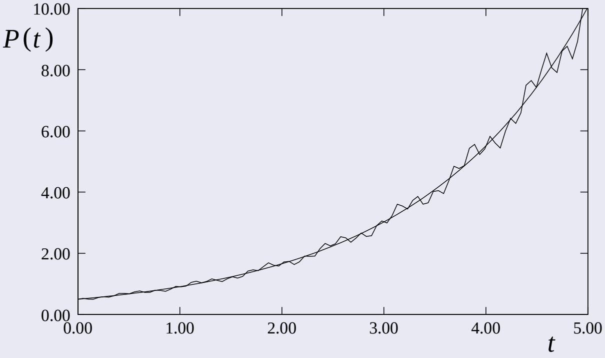
$$\mathrm{E}[P(t)] = \mathrm{E}[P(0)] \cdot e^{\left(\alpha - \frac{\beta^2}{2}\right)t} \cdot \mathrm{E}[e^{(\beta W(t))}] = \mathrm{E}[P(0)] e^{\alpha t} \quad (19)$$

- Therefore, the average value of the interference power grows exponentially, for the specified conditions
- The initial average power, $\mathrm{E}[P(0)]$, should be determined from collected data, or from equipment data sheet, as well as, the parameter, α, that controls the curve

Stochastic Model for the Interference Problem

Stochastic Model for the Interference Problem

- The figure illustrates a stochastic process, that represents the evolution of the interference power with time, as well as, the mean value of the process $\mathrm{E}[P(t)]$, as a continuous exponential curve, for $\alpha = 0.46$



The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- Note that the stochastic process is non-stationary, in the long run, because its average value depends on time
- Besides, the probability distribution of process $P(t)$, that represents the interference, is not Gaussian, as is usually assumed in the computation of the error probability, using the usual theory of stochastic processes
- The stochastic process $W(t)$ results from the combination of several interferers, $W_i(t)$, as the users decide to place calls at the same time, or in the same time interval
- Therefore, inserting Equation 6 into Equation 18, and making $P_0 = P(0)$, results in

$$P(t) = P_0 \cdot e^{(\alpha - \frac{\beta^2}{2})t} \cdot e^{\beta \sum_{i=1}^{N} W_i(t)} = P_0 \cdot e^{(\alpha - \frac{\beta^2}{2})t} \cdot Y(t) \quad (20)$$

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- This clarifies the function of the combination on the interference actual distribution, considering that the first term is a constant, the second term has a deterministic and exponential nature, a function of time that depends on two measurable parameters α and β
- The third term is a function of the interference accumulated in the channel

$$Y(t) = e^{\beta W(t)} = e^{\sum_{i=1}^{N} \beta W_i(t)} = \prod_{i=1}^{N} e^{\beta W_i(t)} \tag{21}$$

and the formula illustrates the exponential nature of the interference on the final distribution

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The Influence of the Combination of Interferers

- In order to simplify the analysis, it is possible to assume that the exponential growth is compensated by an automatic gain control (AGC) subsystem, adjusting the parameters to $\alpha = \frac{\beta^2}{2}$
- This leaves the term $Y(t)$, that is a nonlinear function of the interference sum, which has a Gaussian distribution, if the number of interferers is large, as stated by the Central Limit Theorem
- The following development demonstrates the resulting actual interference distribution

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- Consider that $W(t)$ is a Gaussian random process, because it results from the composition of several effects
- It is possible to obtain the probability distribution of the resulting random process $Y(t)$, using probability density function transformation
- Consider that the process $W(t)$ has probability density function $f_W(w)$, given by

$$f_W(w) = \frac{1}{\sigma_W \sqrt{2\pi}} e^{-\frac{w^2}{2\sigma_W^2}} \tag{22}$$

in which, the mean value of the random process $W(t)$ is given by $m_W = \mathrm{E}[W(t)] = 0$, and the variance is $\sigma_W^2 = \mathrm{E}[W^2(t)]$

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- Because the resulting process $Y(t)$ has distribution $f_Y(y)$, then, it is possible to use the property of probability function transformation, to obtain

$$f_Y(y) = \frac{f_W(w)}{|dy/dw|}, \;\; w = f^{-1}(y) \tag{23}$$

- The derivative of the output process, regarding the input process, in simplified notation, it given by $\frac{dy}{dw} = \beta e^{\beta w}$
- Substituting, one obtains

$$f_Y(y) = \frac{e^{-\frac{w^2}{2\sigma_W^2}}.}{\sigma_W\sqrt{2\pi}|\beta e^{\beta w}|}, \;\; w = \frac{\ln y}{\beta}$$

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- After substituting the inverse function, assuming that β is positive, and that the exponential function is positive definite, produces, after some simplification

$$f_Y(y) = \frac{e^{-\frac{(\ln y)^2}{2\beta^2\sigma_W^2}}}{\beta y \sigma_W \sqrt{2\pi}}, \text{ if } y > 0, \text{ and null in case } y \leq 0 \quad (24)$$

  that represents the Lognormal probability density function, which, different from the Gaussian distribution, is asymmetrical

- The mean value of the random process $Y(t)$, for the resulting distribution, is given by

$$m_Y = \mathrm{E}[Y(t)] = e^{\left[\frac{\beta\sigma_W}{2}\right]} \quad (25)$$

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- One observes that the expected value of process $Y(t)$ grows as an exponential function of the combination of the average value with the standard deviation, adjusted by the parameter β, in relation to the original process $W(t)$, that represents the combination of interferers
- The variance of the random process $Y(t)$ is given by

$$\sigma_Y^2 = \mathrm{E}[(Y(t) - m_Y)^2] = e^{[\beta^2 \sigma_W^2]} \left[e^{(\beta^2 \sigma_W^2)} - 1 \right] \qquad (26)$$

- It is possible to observe that the variance of the combined interference is a function of the product of the exponential of the sum of means and the adjusted original variances, by the exponential of the adjusted variance minus one

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- The probability distribution of the interfering power, $P(t)$, can be found following the same steps

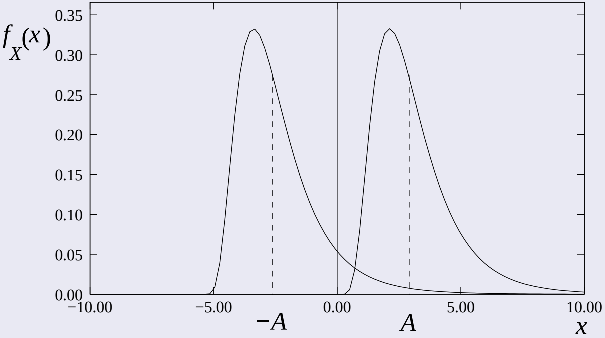
$$f_P(p) = \frac{P_0 e^{-\frac{[\ln p - \ln P_o]^2}{2\beta^2\sigma_W^2}}}{\beta p \sigma_W \sqrt{2\pi}}, \text{ if } p > 0, \text{ and null in case } p \leq 0 \tag{27}$$

- In other words, the random process that represents the interference in a communication system that suffers from an epidemic attack, has a Lognormal probability distribution, instead of the usual assumption of a Gaussian distribution

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- The figure illustrates the probability density function, $f_X(x)$, of a binary received signal, $x(t)$, subject to an interference that has a Lognormal distribution, such as shown in Alencar



The Influence of the Combination of Interferers

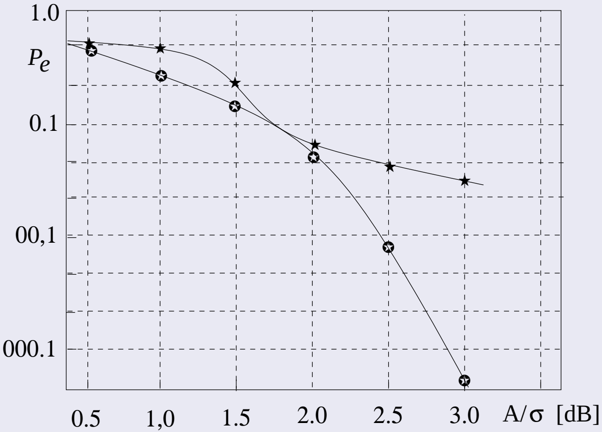
The Influence of the Combination of Interferers

- The derived distribution can be used to compute the symbol or bit error probability for different modulation schemes
- It can also be used to compute, or estimate, the channel capacity or the possible transmissin rate
- The Lognormal probability density function is a heavy tail distribution, that is, it decays slower than an exponential
- Therefore, there is an expected increase in the error probability during the power adaptation process.

The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- The figure illustrates the probability of error for a Binary Phase Shift Keying (BPSK) transmission subject to Gaussian noise (circles), and affected by epidemic interference (stars).



The Influence of the Combination of Interferers

The Influence of the Combination of Interferers

- It can be noted that the epidemic interference, caused by a sudden outburst of signals in the channel increases the error probability, even if all other parameters are kept the same
- The curve in the epidemic interference case remains above the one for the usual Gaussian noise for every level of signal to interference, or noise, power.
- For such a non-stationary process, Itô integration has been used to solve the problem, which, for the defined constraints, resulted in a Lognormal probability density function, that, different from the Gaussian distribution, is asymmetrical
- The long term interference average and variance values follow exponential curves, which depend on certain system parameters

Conclusions

- It is usual to assume stationarity in the analysis of symbol or bit error probability, because the analysis of a non-stationary environment is mathematically complex
- But, in cellular communication systems, the interference is usually dependent on the state of the network, and if the users decide to access the network all of a sudden, the traffic modelling, as well as the interference, is non-stationary
- The current cellular networks use smart communication technologies to regulate the transmitters power as a function of the users' location, in order to limit the interference

Conclusions

- But the control system does not limit the total interference power, such as the one caused by the sudden connection of several cell phones to the air interface
- This article presented a mathematical modelling of the effect of an interference accumulation, caused by a sudden increase of users in a digital cellular system, also called an information outbreak or epidemic interference
- For such a heavy tail distribution, there is an expected increase in the error probability during the adaptation process

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